REPRESENTATION OF BINARY SYSTEMS BY FAMILIES OF BINARY RELATIONS

BY

D. TAMARI AND A. GINZBURG

ABSTRACT

It is proved that an arbitrary binary multiplicative system can be represented by a family of binary relations, using the so called generalized multiplication of relations. Transformations of such representations and existence of a "universal" representation are studied.

Introduction. One considers *families R of binary relations* (f.o.b.r.) over a given class N:

$$r \subset N \times N, R = \{\cdots, r, \cdots\}.$$

The relations r are combined by the *generalized multiplicaton* (g.m.) of binary relations (b.r.) relative to the given family R. This g.m. yields the representation of an arbitrary binary (multiplicative) system (b.s.), associative or not-associative, partial or complete, single or many valued. Indeed, at small expense one can do away with all limitations and extend the validity of the earlier representation theorems [4b] to the most general b.s. M, which is more precisely a couple ($|M|$, T_M), where $|M|$ is an arbitrary class and $T_M \subset |M| \times |M| \times |M|$ an arbitrary ternary relation over $|M|$.

F.o.b.r. with the g.m. can be expressed by a generalization of Brandt's well-known normal multiplication table for groups. Some instances of this generalization have been met in [7]. It is also familiar from the multiplication of ordinary fractions representing positive rational numbers as binary relations over the natural numbers.

Although this approach is very general, it stands the test of useful applications to more specific conventional b.s. and, in particular, to groups.

§1 introduces notations and concepts ;§2 establishes the basic representation theorem in full generality by constructing specific representations (slight extensions of earlier constructions [4b, 6]); §3 considers *transformations* of representations,

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the existence of a *universal* representation, and some *identifications* in the fundamental class N.

Notations in this paper follow [7]. More specific applications are investigated in two other papers [2, 3]. (Cf. also [1, 5]). This series of papers is selfcontained. The results go back to earlier work of the authors (unpublished or stated without proof) mentioned in the bibliography for historical accuracy.

1. **Basic definitions and notations.** $M = (|M|, \cdot) = (|M|, T_M)$ will denote an arbitrary b.s., i.e., a non-empty class $|M| = \{\cdots, a, b, c, \cdots, m, \cdots\}$ with a non-empty ternary relation T_M:

$$(a, b, c) \in T_M \Leftrightarrow c \in ab \text{ in } M.$$

Often we write M instead of $|M|$ and T instead of T_M.

Binary relations associated with the ternary relation T:

$$(a, b, c) \in T \Leftrightarrow b/c \in T^1(a) \Leftrightarrow a/c \in T^2(b) \Leftrightarrow a/b \in T^3(c).$$

Note: $ab = T^1(a)(b) = T^2(b)(a)$.

Projections: $p_iT = \bigcup_{m \in M} T^i(m)$ $(i = 1, 2, 3)$; p_3T is also called the *composability* relation $C = C_M$ of M.

$$\pi_iT = M^i = \{m \mid T^i(m) \neq \varnothing\};\ M^1 = /C,\ M^2 = C/,\ M^3 = \bigcup_{a,b \in M} ab.$$

(All projections are, of course, non-empty.)

$\bigcup_{i=1}^{3} M^i = \mu M$ is the *multiplicative part* of M, $M - \mu M = M^0$ its unessential part.

The concepts of isomorphism, isomorphic systems, and faithful (i.e. isomorphic) representation are defined as usual; those of homomorphism and homomorphic image will not be considered here, except for special situations and, possibly, with some modification.

Given a b.s. M and a family $F = \{\cdots, K_m, \cdots; K_0\}$ of disjoint non-empty classes K_m indexed by M and one further, possibly empty, class K_0 indexed by, say 0, $0 \notin M$, one *inflates* the b.s. M to the b.s. M_F by the following trivial construction:

$$|M_F| = \cup F = \{\cdots, f, g, h, \cdots\} \text{ and } T_{M_F} = \bigcup_{(a,b,c) \in T_M} K_a \times K_b \times K_c, \text{ i.e.,}$$

$$(f, g, h) \in T_{M_F} \Leftrightarrow \exists (a, b, c) \in T_M \mid f \in K_a,\ g \in K_b,\ h \in K_c.$$

F is a partition of $|M_F|$; the corresponding equivalence relation, also noted F, is a congruence and has the property

$$f F g \Rightarrow T^i_{M_F}(f) = T^i_{M_F}(g) \qquad (i = 1, 2, 3).$$

One recuperates M from M_F by *deflation* (*contraction*, *identifications*) mod F: $M_F/F - \{K_0\} \cong M$.

This is an instance of *modified homomorphism* $M_F \to M$; the modification is in the "mapping" of K_0 onto the empty set.

More generally, an equivalence relation D on M with the property

$$a D b \Rightarrow T^i_M(a) = T^i_M(b) \qquad (i = 1, 2, 3)$$

and a, possibly empty, subclass D^0 of equivalence classes mod D in M^0 induce a *deflation* of M to $M^{D,D^0} = M/D - D^0 \; (= (M - \cup D^0)/D)$.

The coarsest deflation equivalence of M, denoted $E = E_M$ is, obviously, defined by

$$a E_M b \Leftrightarrow T^i_M(a) = T^i_M(b) \qquad (i = 1, 2, 3).$$

M^0 is the, possibly empty, equivalence class mod E_M characterized by $T^i_M(m) = \varnothing$ $(i = 1, 2, 3)$.

$$\bar{M} = M/E - \{M^0\} = \{\cdots, \bar{a}, \bar{b}, \bar{c}, \cdots\} \text{ is a b.s. with}$$
$$(\bar{a}, \bar{b}, \bar{c}) \in T_{\bar{M}} \Leftrightarrow (a, b, c) \in T_M \;\big|\; a \in \bar{a}, b \in \bar{b}, c \in \bar{c}\,.$$

$\bar{M}$ is called the *skeleton* or the *essential part* of the b.s. M. If $M^0 = \varnothing$ and all other equivalence classes mod E_M are singletons, then there is an obvious identification of M with $\bar{M} \cong M$. This is, in particular, the case for $\bar{M}$, i.e., $\bar{M} = \bar{\bar{M}}$.

For arbitrary M pick one element m' from each equivalence class mod E_M of μM as its representative and denote the class of representatives M'. Then $M' \subset M$ is a *representative binary subsystem* of M with $M' \cong \bar{M}$ and obvious identification. This justifies calling $\bar{M}$ the "essential part" of M. The remaining elements of each equivalence class mod E_M, if any, may be considered identical copies of m' distinguished from m' and among themselves by special labels (see p. 25). One expresses the same by saying that elements of one and the same equivalence class mod E_M cannot be *separated by multiplicative properties*, i.e, properties of T_M, but elements in distinct equivalence classes can.

Let $R = \{\cdots, r_a, r_b, r_c, \cdots\}$ be a f.o.b.r. over a class $N = N_R$ (R, all r, and of course N non-empty). One defines the *g.m. of relations relative to the family* R by

$$r_c \in r_a \cdot r_b \Leftrightarrow r_c \supset r_a r_b \neq \emptyset,$$

where $r_a r_b$ denotes the usual composition of relations. Thus, in particular,

$$r_a \cdot r_b = \emptyset \Leftrightarrow \{r_a r_b = \emptyset \text{ or there exsists no } r_c \in R \mid r_c \supset r_a r_b \neq \emptyset\}$$

Every f.o.b.r. R becomes with this g.m. a uniquely defined b.s. denoted by the same R and referred to as a b.s. of relations (b.s.o.r.).

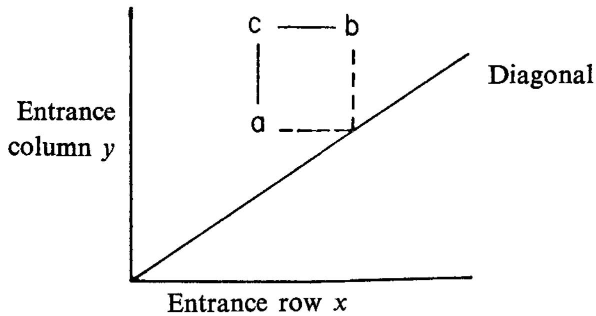
As a matter of fact all f.o.b.r. or, what is now the same, b.s.o.r. appearing in this paper will satisfy

$$r_a \cdot r_b = \emptyset \Leftrightarrow r_a r_b = \emptyset.$$

It follows that $r_a r_b \neq \emptyset \Rightarrow \exists r_c \in R \mid r_a r_b \subset r_c$.

Normal multiplication tables (n.m.t.)

A b.s.o.r. can be expressed by the following generalization [4a] of Brandt's n.m.t



The entrance row and column consist of the elements of N, the *diagonal* of the points $(x; x)$, $x \in N$. If $x/y \in r_a$ the point $(x; y)$ is marked a. The construction of the table ensures that $c \in ab \Leftrightarrow r_c \supset r_a r_b$.

A row and column intersecting on the diagonal are called *corresponding*.

Comparison of families of classes:

Let $F = \{\cdots, F_m, \cdots\}$, $G = \{\cdots, G_m, \cdots\}$ be two families of classes indexed by the same class $M = \{\cdots, m, \cdots\}$. F is *smaller* than G if $F_m \subset G_m$ for all m.

2. The basic representation theorem.

A. THEOREM 1.

Every b.s. can be represented faithfully by a b.s.o.r.

Proof by construction of a particular representation $R_0 = \{\cdots, r_m, \cdots\}$.

The construction of this representation is motivated by the following four roles an element $m \in M$ can play in M:

(1) m may appear as a first (left) factor, i.e., as the first component of a triple of T_M: $m \in M^1$, i.e., $C(m) \neq \emptyset$.

(2) m may appear as a second (right) factor, i.e., as the second component of a triple of T_M: $m \in M^2$, i.e., $C^{-1}(m) \neq \emptyset$.

(3) m may appear as an element in a product, i.e., as the third component of a triple of T_M: $m \in M^3$, i.e., $T^3(m) \neq \emptyset$.

(4) In any case m *must* appear as an element of the class $|M|$, i.e., $m \in |M|$.

Let $N_{R_0} = N_0$ be the union of three distinct copies of $|M| \times |M|$ and two distinct copies of $|M|$:

$$N = \{x_{m_1 m_2}\} \cup \{y_{m_1 m_2}\} \cup \{z_{m_1 m_2}\} \cup \{u_m\} \cup \{v_m\} \; (\forall m, m_1, m_2 \in M).$$

For every $m \in M$ put

$$r_m = r_m^{(1)} \cup r_m^{(2)} \cup r_m^{(3)} \cup r_m^{(4)},$$

where

$$r_m^{(1)} = \{x_{mb}/y_{mb}\}_{b \in C(m)}$$

$$r_m^{(2)} = \{y_{am}/z_{am}\}_{a \in C^{-1}(m)}$$

$$r_m^{(3)} = \{x_{ab}/z_{ab}\}_{a/b \in T^3(m)}$$

$$r_m^{(4)} = \{u_m/v_m\}.$$

Thus, to each role (v) $(v = 1, 2, 3, 4)$ of the element $m \in M$ there corresponds a non-empty relation $r_m^{(v)}$. The correspondence

$$m \to r_m = \bigcup_{v=1}^{4} r_m^{(v)} \text{ is 1-1, since}$$

$$m_1 \neq m_2 \Rightarrow r_{m_1}^{(4)} \neq r_{m_2}^{(4)} \Rightarrow r_{m_1} \neq r_{m_2}.$$

Further, from inspection of the construction, for all a, b, $\in M$:

$$r_a r_b = r_a^{(1)} r_b^{(2)} = \begin{cases} \emptyset \Leftrightarrow ab = \emptyset \\ \{x_{ab}/z_{ab}\} \Leftrightarrow ab \neq \emptyset \end{cases}$$

and

$$c \in ab \Leftrightarrow x_{ab}/z_{ab} \in r_c^{(3)}, \quad \text{i.e.,} \quad \emptyset \neq r_a r_b \subset r_c^{(3)} \subset r_c.$$

Therefore,

$$c \in ab \Leftrightarrow r_c \in r_a \cdot r_b,$$

and, in particular, $ab = \emptyset \Leftrightarrow r_a \cdot r_b = \emptyset$.

Thus, the b.s. of relations R_0 is a faithful representation of M.

Note: According to what was said before, in R_0

$$r_a r_b = \emptyset \Leftrightarrow r_a \cdot r_b = \emptyset.$$

The pair u_m/v_m will be called the *label* of m.

Some economy in the size of the representing relations can always be obtained by discarding the *unnecessary labels* in accordance with an earlier remark about the *necessary labels* as follows:

$$a\, D_M\, b \Leftrightarrow \begin{cases} a = b & \text{or} \\ T^1(a) = T^1(b) = T^2(a) = T^2(b) = \emptyset, & T^3(a) = T^3(b). \end{cases}$$

Notice that $D_M \subset E_M$ and that M^0 is an equivalence class of D_M. Choose an arbitrary element m from every equivalence class of D_M except from M^0, and delete $r_m^{(4)}$ from r_m, thus distinguishing this element from others in its class by the absence of $r_m^{(4)}$. The family of relations obtained in this way from R_0 will be denoted by R_I. R_I is still a faithful representation of M.

Note that R_I depends on the choice of particular elements (representatives) from the classes $\neq M^0$ of D_M. One can pass from one such choice to another by a product of disjoint transpositions of the elements of M. Since the permuted elements play exactly the same multiplicative role in M, all choices are equivalent. Therefore, we can assume, without loss of generality, for various f.o.b.r. representing the same M a fixed choice of representatives.

If every equivalence class mod D_M consists of a singleton and $M^0 = \emptyset$, then R_I is obtained from R_0 by deleting all $r_m^{(4)}$.

The representation R_I is smaller than R_0. Besides having the properties of R_0

$$1)\ r_a r_b \neq \emptyset \Rightarrow \exists r_c \supset r_a r_b, \quad 2)\ r_a r_b \cap r_c \neq \emptyset \Rightarrow r_a r_b \subset r_c,$$

R_I is also 3) *minimal* (relative to the comparison of families of classes defined on p. 24) among the faithful representations of M.

Definition: A b.s.o.r. satisfying properties 1), 2), 3) will be called *reduced*.

EXAMPLES

1) Let M be given by the multiplication table

	a	b	c
a			
b			
c		a, b	

$R_0 = \{r_a, r_b, r_c\}, \quad r_a = \{x_{cc}/z_{cc}, \quad u_a/v_a\}, \quad r_b = \{x_{cc}/z_{cc}, \quad u_b/v_b\},$

$r_c = \{x_{cc}/y_{cc}, \quad y_{cc}/z_{cc}, \quad u_c/v_c\}$

The equivalence classes mod D_M are $\{a, b\}$ and $\{c\}$. An $R_I = \{r'_a, r'_b, r'_c\}$ is, e.g.: $r'_a = \{x_{cc}/z_{cc}\}$, $r'_b = r_b$, $r'_c = \{x_{cc}/y_{cc}, y_{cc}/z_{cc}\}$.

2)

	a	b	c
a		a, b	
b	a, b, c		b
c	a		c

$R_I = \{r_a, r_b, r_c\}$,

$$r_a = \{x_{ab}/y_{ab}, y_{ba}/z_{ba}, y_{ca}/z_{ca}, x_{ab}/z_{ab}, x_{ba}/z_{ba}, x_{ca}/z_{ca}\}$$

$$r_b = \{x_{ba}/y_{ba}, x_{bc}/y_{bc}, y_{ab}/z_{ab}, x_{ab}/z_{ab}, x_{ba}/z_{ba}, x_{bc}/z_{bc}\}$$

$$r_c = \{x_{ca}/y_{ca}, x_{cc}/y_{cc}, y_{bc}/z_{bc}, y_{cc}/z_{cc}, x_{ba}/z_{ba}, x_{cc}/z_{cc}\}$$

B. *Another representation of M.*

Let N be the union of three distinct copies of $|M|$:

$$N = \{x_m\} \cup \{y_m\} \cup \{z_m\} \quad (m \in M)$$

Consider the family of relations $R_{II} = \{\cdots, r_m, \cdots\}$ where:

(1) $$r_m = \{x_m/y_m\} \cup \{y_i/z_m\}_{i \in C^{-1}(m)} \cup \{x_i/z_j\}_{i/j \in T^3(m)}$$

R_{II} is isomorphic to M. Indeed, $m_1 \neq m_2 \Rightarrow r_{m_1} \neq r_{m_2}$. Furthermore

$$c \in ab \Leftrightarrow a \in C^{-1}(b) \text{ and } x_a/z_b \in r_c,$$

hence

$$c \in ab \Leftrightarrow (r_a r_b = \{x_a/y_a\}\ \{y_a/z_b\} = \{x_a/z_b\} \subset r_c) \Leftrightarrow r_c \in r_a \cdot r_b.$$

Similarly to the reduction of R_0 to R_I one can reduce R_{II} by suppressing unnecessary labels of the form x_m/y_m. Indeed, after choosing as before a representative m' from every equivalence class of D_M, except from M^0, $x_{m'}/y_{m'}$ can be deleted from the corresponding relation $r_{m'}$, provided that $C(m') = \varnothing$ (otherwise $x_{m'}/y_{m'}$ is needed to represent m' as a left factor).

The b.s. of relations R_{III} obtained in this way from R_{II} is still a faithful representation of M and is reduced.

Examples

For example 2 $R_{II} = R_{III} = \{r_a, r_b, r_c\}$ with

$$r_a = \{x_a/y_a,\ y_b/z_a,\ y_c/z_a,\ x_a/z_b,\ x_b/z_a,\ x_c/z_a\}$$

$$r_b = \{x_b/y_b,\ y_a/z_b,\ x_a/z_b,\ x_b/z_a,\ x_b/z_c\}$$

$$r_c = \{x_c/y_c,\ y_b/z_c,\ y_c/z_c,\ x_b/z_a,\ x_c/z_c\}$$

and the n.m.t.:

	x_a	x_b	x_c	y_a	y_b	y_c	z_a	z_b	z_c
z_c		b	c		c	c			
z_b	a,b			b					
z_a		a,b,c	a		a	a			
y_c			c						
y_b		b							
y_a	a								
x_c									
x_b									
x_a									

3) M:

	a	b	c	d
a	a	c,d		
b				
c				
d				

$R_{II} = \{r_a, r_b, r_c, r_d\}$, where

$r_a = \{x_a/y_a,\ y_a/z_a,\ x_a/z_a\}$, $r_b = \{x_b/y_b,\ y_a/z_b\}$, $r_c = \{x_c/y_c,\ x_a/z_b\}$,

$r_d = \{x_d/y_d,\ x_a/z_b\}$

The equivalence classes mod D_M are $\{a\}$, $\{b\}$, $\{c,d\}$.
An R_{III} is, e.g.,

$$r'_a = r_a, \quad r'_b = \{y_a/z_b\}, \quad r'_c = \{x_a/z_b\},\ r'_d = r_d.$$

REMARK. Instead of (1) one can put

$$r'_m = \{x_m/y_i\}_{i\in C(m)} \cup \{y_m/z_m\} \cup \{x_i/z_j\}_{i/j\in T^3(m)}$$

and obtain a representation R'_{II} "dual" to R_{II}.

3. **Transforms of representations.** Let N and N^1 be classes, $R = \{\cdots, r_a, \cdots\}$ an indexed family of non-empty relations over N and $\alpha \subset N \times N^1$.

$$R^\alpha = \{\cdots, r^\alpha_m, \cdots\}, \text{ where } r^\alpha_m = \alpha^{-1} r_m \alpha,$$

will be called the *transform* of R by α.

In particular, if α is a mapping the corresponding transform will be called a *mapping transform.*

R^α will be a family of non-empty relations if and only if for every m

$$(/\alpha \times /\alpha) \cap r_m \neq \varnothing.$$

PROPOSITION 1. *For any b.s. M, there exists a mapping α such that $R_{II} = R^\alpha_0$ and $R_{III} = R^\alpha_I$.*

Proof by construction of α.

$$N = \{x_{ab}\} \cup \{y_{ab}\} \cup \{z_{ab}\} \cup \{u_a\} \cup \{v_a\} \qquad (a, b \in M)$$

$$N^1 = \{x_a\} \cup \{y_a\} \cup \{z_a\} \qquad (a \in M)$$

$$\alpha = \{x_{ab}/x_a\} \cup \{y_{ab}/y_a\} \cup \{z_{ab}/z_b\} \cup \{u_a/x_a\} \cup \{v_a/y_a\} \qquad (a, b \in M)$$

For any $r_m \in R_0$ one computes that

$$\alpha^{-1} r_m \alpha = r^\alpha_m = \{x_m/y_m\} \cup \{y_i/z_m\}_{i\in C^{-1}(m)} \cup \{x_i/z_j\}_{i/j\in T^3(m)} = r'_m \in R_{II}$$

Hence $R_{II} = R^\alpha_0$. Similarly, $R_{III} = R^\alpha_I$, because the choice of the same representatives from the equivalence classes of D_M ensures that

$$\alpha^{-1} r_m \alpha = r'_m \text{ for every } r_m \in R_I \text{ and the corresponding } r'_m \in R_{III}.$$

REMARK. $R'_{II} = R^{\alpha'}_0$, where

$$\alpha' = \{x_{ab}/x_a\} \cup \{y_{ab}/y_b\} \cup \{z_{ab}/z_b\} \cup \{u_a/y_a\} \cup \{v_a/z_a\} \qquad (a, b \in M).$$

Compound transform

Let $A = \{\alpha\}$ be a class of relations $\alpha \subset N \times N^1$. The (compound) transform of R by A is $R^A = \{\cdots, r_m^A, \cdots\}$, where

$$r_m^A = \bigcup_{\alpha \in A} r_m^\alpha = \bigcup_{\alpha \in A} \alpha^{-1} r_{m^\alpha}.$$

The following theorem shows that R_I is in a certain sense a *universal* representation.

THEOREM 2. *For any representation of M by a reduced b.s.o.r. $S = \{\cdots, s_m, \cdots\}$ there exist an R_I representing M and a family $A = \{\alpha\}$ such that $S = R_I^A$.*

Proof by construction of a particular R_I and a particular family A. The fact that S is reduced implies some choice of unlabeled elements of M; in accordance to our convention R_I belongs to the same choice.

Assume $\sigma/\tau \in s_m \in S$. σ/τ must represent m in at least one of its four possible roles in M:

1) $(m,b,c) \in T \,\&\, \exists \zeta \in N_S \,|\, \tau/\zeta \in s_b,\ \sigma/\zeta \in s_c$; in this case put $\alpha = \{x_{mb}/\sigma, y_{mb}/\tau\}$;

2) $(a,m,c) \in T \,\&\, \exists \xi \in N_S \,|\, \xi/\sigma \in s_a,\ \xi/\tau \in s_c$; in this case put $\alpha = \{y_{am}/\sigma, z_{am}/\tau\}$;

3) $(a,b,m) \in T \,\&\, \exists \eta \in N_S \,|\, \sigma/\eta \in s_a,\ \eta/\tau \in s_b$; in this case put $\alpha = \{x_{ab}/\sigma, z_{ab}/\tau\}$;

4) m is a labeled element and σ/τ is the label; in this case put $\alpha = \{u_m/\sigma, v_m/\tau\}$

In all four cases one computes $\{\sigma/\tau\} = r_m^\alpha$. A is the collection of all these α, and, therefore, for all m,

$$s_m = \bigcup_{\alpha \in A} r_m^\alpha.$$

The b.r. ρ between representations of M with a fixed labeling

$$R_1 \rho R_2 \Leftrightarrow \exists A \,|\, R_2 = R_1^A \quad (A = \{\alpha\}),$$

is obviously a quasi-ordering (reflexive and transitive). Thus, the *universality* o R_I means that R_I is "greatest" in the class of reduced representations of M with that labeling. The possible (quasi-) order relations in the class of representations of a b.s. M deserve further investigation.

Some simple mapping transforms: identifications.

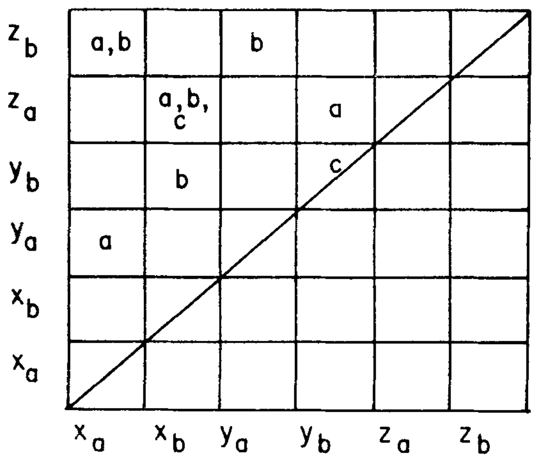
Let M be represented by R_{III} and consider the following "identifications":

1) For $C(a) = C(b)$: Put $y_a = y_b$, i.e., $N\alpha = N - \{y_b\}$, α is the identity on $N - \{y_b\}$ and $y_b\alpha = y_a$.

2) For $e \in C(e)$ and $(\forall m \in C(e))(em = m)$: Put $x_e = y_e$, i.e., $N\alpha = N - \{y_e\}$, α is the identity on $N - \{y_e\}$ and $y_e\alpha = x_e$.

3) For $e \in C(e)$, $(\forall m_1 \in C^{-1}(e))(m_1 e = m_1)$, $(\forall m_2 \in C(e))(em_2 = m_2)$ and $C^{-1}(e) \times C(e) \subset C$: Put $x_e = y_e = z_e$.

In all cases R^{α}_{III} remains a faithful representation of M. In example 2 (p. 28) conditions 1) and 3) hold simultaneously; the identifications $y_b = x_c = y_c = z_c$ yield the n.m.t.:



Corollary 1. For a monad (groupoid) all y_m can be identified (by condition 1)). Therefore, for a faithful representation of a monad M by a f.o.b.r. card $N \leqq 2$ card $M + 1$ suffices; and if M has an identity, 2 card $M - 1$.

4) A further identification for a monad M with an identity e and elements a, b such that $ab = ba = e$ and $(\forall m_1, m_2 \in M)((m_1 a)(bm_2) = m_1 m_2)$: Put $z_a = x_b$.

Corollary 2. For a group G card $N =$ card G suffices.

Proof. Put in 4) $a = g$, $b = g^{-1}$ for all $g \in G$; by associativity $(g_i g)(g^{-1} g_j) = g_i g_j$ for all $g_i, g_j \in G$; identifying $z_g = x_{g^{-1}}$, one obtains $N = \{\cdots, x_g, \cdots\}$. This gives the well known n.m.t. for groups (see, e.g., [4a, 8 p.4]). Further reductions of card N for groups are treated elsewhere [2].

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State University of New York at Buffalo,
Buffalo, N.Y.
Technion—Israel Institute of Technology,
Haifa